

CONNECTIONS BETWEEN THE LIBERATION OF PROJECTIONS AND ITS COUNTERPART FOR SYMMETRIES

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ABSTRACT. We present here some connections between the liberation process for projections $(P, Q) \mapsto (P, U_t Q U_t^*)$ and its counterpart $(R, S) \mapsto (R, U_t S U_t^*)$ for symmetries when the projections $\{P, Q\}$ and the symmetries $\{R, S\}$ are associated, where U_t is a free unitary Brownian motion freely independent from $\{P, Q\}$ (and so $\{R, S\}$). We relate the moments of their actions on the operators $X_t := P U_t Q U_t^*$ and $Y_t := U_t R U_t^* S$ and use this to prove a relationship between the corresponding spectral measures (hereafter μ_t and ν_t). On the other hand, we focus in the process of unitary random variables Y_t in the case of arbitrary trace values $\tau(R), \tau(S)$. More precisely, we use stochastic calculus to derive a partial differential equation (PDE for short) for its Herglotz transform and use it to develop subordination results in terms of Löwner equations. The paper is closed with an improved proof of $i^* (\mathbb{C}P + \mathbb{C}(I - P); \mathbb{C}Q + \mathbb{C}(I - Q)) = -\chi_{orb}(P, Q)$ as an application.

1. INTRODUCTION

Let (\mathcal{A}, τ) be a W^* -probability space and $U_t, t \in [0, \infty)$ a free unitary Brownian motion in (\mathcal{A}, τ) with $U_0 = \mathbf{1}$. For a given pair of orthogonal projections $\{P, Q\}$ in \mathcal{A} that are freely independent from $(U_t)_{t \geq 0}$, the so-called liberation process $(P, Q) \mapsto (P, U_t Q U_t^*)$ was introduced in [14] in relation with the free entropy and the free Fisher information. We look here to its counterpart $(R, S) \mapsto (R, U_t S U_t^*)$ when $\{R, S\}$ are two symmetries associated to $\{P, Q\}$ via $R = 2P - \mathbf{1}, S = 2Q - \mathbf{1}$. It is known, as consequence of the asymptotic freeness of P and $U_t Q U_t^*$, that the pair $(R, U_t S U_t^*)$ tends, as $t \rightarrow \infty$, to $(R, U S U^*)$ where U is a Haar unitary free from $\{R, S\}$ and hence $R, U S U^*$ are free (see [13]). The connection between the two liberation processes can be understood by looking to the relationship between their actions on the operators $P U_t Q U_t^*$ and $R U_t S U_t^*$. Thus, we mainly investigate this relationship in what follow. The purpose of this study is to investigate the motivating question of proving $i^* = -\chi_{orb}$ for two projections. An heuristic argument for this question in [10, Section 3.2] supports that the equality holds. Recently, Collins and Kemp [2] gave a proof of the equality for two projections with $\tau(P) = \tau(Q) = 1/2$. This result was subsequently proved by Izumi and Ueda [11]. They go further and use a subordination relation to give some partial results for the general case.

In the present paper, we give an improved assertion of the result in [11] based on a similar subordination relation. To this end, we study the dynamic of the unitary process $Y_t = U_t R U_t^* S$. More precisely, we use stochastic calculus to derive a system of ODEs for its sequence of moments. The obtained system is transformed into a PDE for the Herglotz transform (hereafter $H(t, z)$) of its corresponding spectral measure ν_t . In particular, we supply a full description of the measure of the steady-state solution. Then, we develop a

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theory of subordination for the process Y_t akin to [11] and obtain an explicit computation of the unique subordinate family. This allows us, in particular, to show that the boundary of its range is at a positive distance from ± 1 and use it to prove a certain regularity condition for the obtained subordination relation. On the other hand, we generalize the approach used in [5] relating the moments of $X_t = PU_tQU_t^*$ and those of $Y_t = RU_tSU_t^*$ to the case of two arbitrary projections. The obtained relation is then transformed into a relationship between their corresponding measure μ_t and ν_t . Finally, we obtain a partial result for the identity $i^* = -\chi_{orb}$ in the case of arbitrary values of traces $\tau(P), \tau(Q)$ as application of the tools developed in this paper.

2. ANALYSIS OF THE SPECTRAL MEASURE OF Y_t

2.1. Sequence of moments. Let $R, S \in \mathcal{A}$ be two symmetries with $\tau(R) = \alpha$ and $\tau(S) = \beta$ and $U_t, t \in [0, \infty)$ a free unitary Brownian motion freely independent from $\{R, S\}$. Let ν_t be the spectral distribution of the unitary process $Y_t = RU_tSU_t^*$ on \mathbb{T} (the set of complex numbers with modulus one). Our goal here is to derive a system of ODEs satisfied by the sequence of moments of ν_t via free stochastic calculus.

Proposition 2.1. *Let $f_n(t) := \tau[(RU_tSU_t^*)^n]$ $n \geq 1, t \geq 0$, then*

$$\partial_t f_1 = -f_1 + \alpha\beta,$$

$$\partial_t f_n = -nf_n - n \sum_{k=1}^{n-1} f_k f_{n-k} + \begin{cases} n^2 \alpha \beta & \text{if } n \text{ is odd} \\ n^2 \frac{\alpha^2 + \beta^2}{2} & \text{if } n \text{ is even} \end{cases}, \quad n \geq 2$$

where $\alpha = \tau(R)$ and $\beta = \tau(S)$.

Proof. Let $A_t = RU_tSU_t^*$, then using Ito's formula, we have

$$d(A_t^n) = \sum_{k=1}^n A_t^{k-1} dA_t A_t^{n-k} + \sum_{1 \leq j < k \leq n} A_t^{j-1} dA_t A_t^{k-j-1} dA_t A_t^{n-k}.$$

Taking the trace in both sides and use the trace property, we get

$$\tau[d(A_t^n)] = \sum_{k=1}^n \tau[A_t^{n-1} dA_t] + \sum_{1 \leq j < k \leq n} \tau[A_t^{n-(k-j)-1} dA_t A_t^{k-j-1} dA_t].$$

The first summands do not depend on the summation variable k , while the second summands depend on the summation variable j, k only through their difference $k - j$. Then re-indexing by $l = k - j$, we get

$$\tau[d(A_t^n)] = n\tau[A_t^{n-1} dA_t] + \sum_{l=1}^{n-1} \sum_{1 \leq j < k \leq n, k-j=l} \tau[A_t^{n-l-1} dA_t A_t^{l-1} dA_t].$$

Since the number of pairs (j, k) such that $k - j = l$ for fixed l is equal to $n - l$, then the second summation becomes

$$\sum_{l=1}^{n-1} (n-l) \tau[A_t^{n-l-1} dA_t A_t^{l-1} dA_t]. \quad (2.1)$$

This sum rewrites, after re-indexing $k = n - l$, as

$$\sum_{k=1}^{n-1} k \tau [A_t^{k-1} dA_t A_t^{n-k-1} dA_t]. \quad (2.2)$$

Using the trace property and adding the summations (2.1) and (2.2), we get

$$\sum_{k=1}^{n-1} (n - k + k) \tau [A_t^{n-k-1} dA_t A_t^{k-1} dA_t] = n \sum_{k=1}^{n-1} \tau [A_t^{n-k-1} dA_t A_t^{k-1} dA_t].$$

Thus, we have

$$\tau [d(A_t^n)] = n \tau [A_t^{n-1} dA_t] + \frac{n}{2} \sum_{k=1}^{n-1} \tau [A_t^{n-k-1} dA_t A_t^{k-1} dA_t]. \quad (2.3)$$

Now since R and S are independent from t , the free Ito's formula implies

$$\begin{aligned} dA_t &= R d(R_t S U_t^*) = R(dU_t) S U_t^* + R U_t d(S U_t^*) + R(dU_t) d(S U_t^*) \\ &= R(dU_t) S U_t^* + R U_t S(dU_t^*) + R(dU_t) S(dU_t^*). \end{aligned}$$

But, since

$$dU_t = i U_t dB_t - \frac{1}{2} U_t dt \quad \text{and} \quad dU_t^* = -i dB_t U_t^* - \frac{1}{2} U_t^* dt.$$

Then substituting these equations in the expression of dA_t we get

$$dA_t = R(i U_t dB_t - \frac{1}{2} U_t dt) S U_t^* + R U_t S(-i dB_t U_t^* - \frac{1}{2} U_t^* dt) + R(i U_t dB_t - \frac{1}{2} U_t dt) S(-i dB_t U_t^* - \frac{1}{2} U_t^* dt).$$

The first two terms simplify to

$$i R U_t dB_t S U_t^* - i R U_t S dB_t U_t^* - R U_t S U_t^* dt = i R U_t dB_t S U_t^* - i R U_t S dB_t U_t^* - A_t dt$$

while the last term is reduced to

$$R(i U_t dB_t) S(-i dB_t U_t^*) = R U_t dB_t S dB_t U_t^* = R U_t \tau(S) U_t^* dt = \beta R dt$$

Thus, we have

$$dA_t = i R U_t dB_t S U_t^* - i R U_t S dB_t U_t^* + (\beta R - A_t) dt. \quad (2.4)$$

So that,

$$A_t^{n-1} dA_t = i A_t^{n-1} R U_t dB_t S U_t^* - i A_t^{n-1} R U_t S dB_t U_t^* + A_t^{n-1} (\beta R - A_t) dt.$$

Since the trace of a stochastic integral is zero, then the first term in equation (2.3) is given by

$$\tau(A_t^{n-1} dA_t) = \tau[A_t^{n-1} (\beta R - A_t)] dt = [\beta \tau(A_t^{n-1} R) - \tau(A_t^n)] dt.$$

Using the trace property and the relations $R^2 = S^2 = U_t U_t^* = 1$, we have $\tau(A_t^{n-1} R) = \tau(R) = \alpha$ if n is odd and $\tau(A_t^{n-1} R) = \tau(S) = \beta$ otherwise.

Hence, the first term in equation (2.3) is equal to

$$n \tau(A_t^{n-1} dA_t) = \begin{cases} [n \beta^2 - n \tau(A_t^n)] dt & \text{if } n \text{ is even} \\ [n \beta \alpha - n \tau(A_t^n)] dt & \text{otherwise} \end{cases}. \quad (2.5)$$

For the second term in equation (2.3), we shall use the following result.

Lemma 2.2. *Let*

$$dZ_t = iRU_t dB_t SU_t^* - iRU_t S dB_t U_t^*. \quad (2.6)$$

Then

$$dt dZ_t = dZ_t dt = (dt)^2 = 0$$

and for any adapted process V_t , we have

$$dZ_t V_t dZ_t = [2R\tau(RV_t) - 2A_t\tau(A_t V_t)]dt. \quad (2.7)$$

Proof. The first statement is a consequence of Itô rules since Z_t is a stochastic integral. For the last, we expand

$$\begin{aligned} dZ_t V_t dZ_t &= (iRU_t dB_t SU_t^* - iRU_t S dB_t U_t^*) V_t (iRU_t dB_t SU_t^* - iRU_t S dB_t U_t^*) \\ &= -RU_t dB_t SU_t^* V_t RU_t dB_t SU_t^* + RU_t dB_t SU_t^* V_t RU_t S dB_t U_t^* + RU_t S dB_t U_t^* V_t RU_t dB_t SU_t^* \\ &\quad - RU_t S dB_t U_t^* V_t RU_t S dB_t U_t^*. \end{aligned}$$

Applying the Itô rule

$$dB_t V_t dB_t = \tau(V_t)dt$$

to each of these terms yields

$$\begin{aligned} dZ_t V_t dZ_t &= -RU_t \tau(SU_t^* V_t RU_t) SU_t^* dt + RU_t \tau(SU_t^* V_t RU_t S) U_t^* dt + RU_t S \tau(U_t^* V_t RU_t) SU_t^* dt \\ &\quad - RU_t S \tau(U_t^* V_t RU_t S) U_t^* dt. \end{aligned}$$

Using the trace property and the relations $S^2 = U_t U_t^* = 1$, $A_t = RU_t SU_t^*$, we get

$$dZ_t V_t dZ_t = -A_t \tau(A_t^* V_t) dt + R \tau(V_t R) dt + R \tau(V_t R) dt - A_t \tau(A_t^* V_t) dt$$

which simplifies to give the equality (2.7). \square

It follows from (2.4) and (2.6) that for $n \geq 2$ and $k \in \{1, \dots, n-1\}$,

$$A_t^{n-k-1} dA_t A_t^{k-1} dA_t = A_t^{n-k-1} [dZ_t + (\beta R - A_t)dt] A_t^{k-1} [dZ_t + (\beta R - A_t)dt]$$

which expands into four terms. But by use of lemma 2.2, the only surviving term is

$$A_t^{n-k-1} dZ_t A_t^{k-1} dZ_t = A_t^{n-k-1} [2R\tau(RA_t^{k-1}) - 2A_t\tau(A_t^k)]dt.$$

Taking the trace, we get

$$\tau(A_t^{n-k-1} dA_t A_t^{k-1} dA_t) = [2\tau(RA_t^{k-1})\tau(RA_t^{n-k-1}) - 2\tau(A_t^k)\tau(A_t^{n-k})]dt$$

Using the same consideration leading to (2.1) and the fact that if n is even then $k, n-k$ have the same parity and if n is odd then $k, n-k$ have opposite parity, we have

$$\tau(A_t^{n-k-1} dA_t A_t^{k-1} dA_t) = \begin{cases} (2\alpha^2 - 2\tau(A_t^k)\tau(A_t^{n-k}))dt & \text{if } n \text{ is even and } k \text{ is odd} \\ (2\beta^2 - 2\tau(A_t^k)\tau(A_t^{n-k}))dt & \text{if } n \text{ is even and } k \text{ is even} \\ (2\alpha\beta - 2\tau(A_t^k)\tau(A_t^{n-k}))dt & \text{if } n \text{ is odd and } k \text{ is odd} \\ (2\alpha\beta - 2\tau(A_t^k)\tau(A_t^{n-k}))dt & \text{if } n \text{ is odd and } k \text{ is even} \end{cases}$$

Hence, the second term in equation (2.3) is equal to

$$\frac{n}{2} \sum_{k=1}^{n-1} \tau(A_t^{n-k-1} dA_t A_t^{k-1} dA_t) = \begin{cases} \left(-n \sum_{k=1}^{n-1} \tau(A_t^k) \tau(A_t^{n-k}) + \frac{n^2}{2} \alpha^2 + \frac{n(n-2)}{2} \beta^2 \right) dt & \text{if } n \text{ is even} \\ \left(-n \sum_{k=1}^{n-1} \tau(A_t^k) \tau(A_t^{n-k}) + n(n-1) \alpha \beta \right) dt & \text{if } n \text{ is odd} \end{cases}$$

which simplifies to

$$\frac{n}{2} \sum_{k=1}^{n-1} \tau(A_t^{n-k-1} dA_t A_t^{k-1} dA_t) = -n \sum_{k=1}^{n-1} \tau(A_t^k) \tau(A_t^{n-k}) + \begin{cases} \left(\frac{n^2}{2} \alpha^2 + \frac{n(n-2)}{2} \beta^2 \right) dt & \text{if } n \text{ is even} \\ (n(n-1) \alpha \beta) dt & \text{if } n \text{ is odd} \end{cases} \quad (2.8)$$

and hence the desired assertions follows after summing (2.5) and (2.8). \square

2.2. The Herglotz transform of ν_t . Here, we derive a PDE governing the Herglotz transform of the spectral measure ν_t :

$$H(t, z) := \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\nu_t(\zeta) = 1 + 2 \sum_{n \geq 1} f_n(t) z^n.$$

Recall that, this is an analytic function on \mathbb{D} (the open unit disc of \mathbb{C}).

Proposition 2.3. *The function $H(t, z)$ satisfies the PDE*

$$\partial_t H + \frac{z}{2} \partial_z H^2 = \frac{2z(\alpha z^2 + 2\beta z + \alpha)(\beta z^2 + 2\alpha z + \beta)}{(1 - z^2)^3}. \quad (2.9)$$

Proof. By direct calculation from Proposition 2.1, we have

$$\begin{aligned} \partial_t H &= 2 \sum_{n \geq 1} \partial_t f_n(t) z^n \\ &= -2 \sum_{n \geq 1} n f_n z^n - 2 \sum_{n \geq 1} n \sum_{k=1}^{n-1} f_k f_{n-k} z^n + (\alpha^2 + \beta^2) \sum_{n \geq 1, n \text{ even}} n^2 z^n + 2\alpha\beta \sum_{n \geq 1, n \text{ odd}} n^2 z^n \\ &= -z \partial_z H - 2 \sum_{k \geq 1} f_k z^k \sum_{n \geq k+1} n f_{n-k} z^{n-k} + 4(\alpha^2 + \beta^2) \frac{z^2(1 + z^2)}{(1 - z^2)^3} + 2\alpha\beta z \frac{1 + 6z^2 + z^4}{(1 - z^2)^3} \\ &= -z \partial_z H - 4 \frac{H - 1}{2} \sum_{n \geq k+1} n f_{n-k} z^{n-k} + \frac{2z(\alpha z^2 + 2\beta z + \alpha)(\beta z^2 + 2\alpha z + \beta)}{(1 - z^2)^3} \\ &= -z H \partial_z H + \frac{2z(\alpha z^2 + 2\beta z + \alpha)(\beta z^2 + 2\alpha z + \beta)}{(1 - z^2)^3}. \end{aligned}$$

\square

2.3. Steady-state solution. As mentioned in the Introduction, it is known from the asymptotic freeness of P and $U_t Q U_t^*$ that

Proposition 2.4. *The spectral measure ν_t of $R U_t S U_t^*$ converges weakly, as $t \rightarrow \infty$, to the free multiplicative convolution of the spectral measures of R and $U S U^*$, where $U \in \mathcal{A}$ is a Haar unitary operator free from $\{R, S\}$.*

We will see this directly from the PDE (2.9). Let $H(\infty, \cdot)$ be the state solution of (2.9), then it satisfies

$$\partial_z H^2 = \frac{4(\alpha z^2 + 2\beta z + \alpha)(\beta z^2 + 2\alpha z + \beta)}{(1 - z^2)^3}.$$

After integration and taking into account $H(\infty, 0) = 1$, we get

$$H(\infty, z) = \sqrt{1 + 4z \frac{\alpha\beta(1+z)^2 + (\alpha - \beta)^2 z}{(1 - z^2)^2}} \quad (2.10)$$

where the principal branch of the square root is taken. On the other hand, the next technical proposition gives an explicit calculation for the Herglotz transform of $\nu_R \boxtimes \nu_S$.

Proposition 2.5. *Let $\mu = \frac{1+\alpha}{2}\delta_1 + \frac{1-\alpha}{2}\delta_{-1}$ and*

$$\nu = \left(\frac{1+\alpha}{2}\delta_1 + \frac{1-\alpha}{2}\delta_{-1} \right) \boxtimes \left(\frac{1+\beta}{2}\delta_1 + \frac{1-\beta}{2}\delta_{-1} \right)$$

for $\alpha, \beta \in (-1, 1]$. Then the Herglotz transform of ν is given by

$$H_\nu(z) = H(\infty, z) = \sqrt{1 + 4z \frac{\alpha\beta(1+z)^2 + (\alpha - \beta)^2 z}{(1 - z^2)^2}}.$$

Proof. Using the analytic machinery for multiplicative convolution (see [8]), we have

$$\begin{aligned} \psi_\mu(z) &= \frac{z(z + \alpha)}{1 - z^2}, \\ \chi_\mu(z) &= \frac{-\alpha \pm \sqrt{\alpha^2 + 4z(z + 1)}}{2(z + 1)}, \\ S_\mu(z) &= \frac{-\alpha \pm \sqrt{\alpha^2 + 4z(z + 1)}}{2z}. \end{aligned}$$

So that

$$\begin{aligned} S_\nu(z) &= \frac{\left(-\alpha \pm \sqrt{\alpha^2 + 4z(z + 1)}\right) \left(-\beta \pm \sqrt{\beta^2 + 4z(z + 1)}\right)}{4z^2}, \\ \chi_\nu(z) &= \frac{\left(-\alpha \pm \sqrt{\alpha^2 + 4z(z + 1)}\right) \left(-\beta \pm \sqrt{\beta^2 + 4z(z + 1)}\right)}{4z(z + 1)}, \end{aligned}$$

and ψ_ν satisfies

$$\frac{\left(-\alpha \pm \sqrt{\alpha^2 + 4\psi_\nu(\psi_\nu + 1)}\right) \left(-\beta \pm \sqrt{\beta^2 + 4\psi_\nu(\psi_\nu + 1)}\right)}{4\psi_\nu(\psi_\nu + 1)} = z.$$

Letting $\varphi_\nu = \psi_\nu(\psi_\nu + 1)$, we get $\psi_\nu = (-1 \pm \sqrt{1 + 4\varphi_\nu})/2$ and since the Herglotz transform has a positive real part, $H_\nu = \sqrt{1 + 4\varphi_\nu}$ where φ_ν is given by

$$\frac{\left(-\alpha \pm \sqrt{\alpha^2 + 4\varphi_\nu}\right) \left(-\beta \pm \sqrt{\beta^2 + 4\varphi_\nu}\right)}{4\varphi_\nu} = z.$$

Or equivalently

$$-\alpha \pm \sqrt{\alpha^2 + 4\varphi_\nu} = z \left(\beta \pm \sqrt{\beta^2 + 4\varphi_\nu} \right).$$

Rearranging this last equality and raising it to the square, we get

$$\alpha^2 + 4\varphi_\nu + z^2(\beta^2 + 4\varphi_\nu) - (\alpha + \beta z)^2 = 2z\sqrt{(\alpha^2 + 4\varphi_\nu)(\beta^2 + 4\varphi_\nu)}.$$

So we raise it to the square once again, to get

$$[\alpha^2 + 4\varphi_\nu + z^2(\beta^2 + 4\varphi_\nu) - (\alpha + \beta z)^2]^2 = 4z^2(\alpha^2 + 4\varphi_\nu)(\beta^2 + 4\varphi_\nu).$$

Which simplifies to

$$2(1 - z^2)^2\varphi_\nu + [(1 - z^2)(\alpha^2 - \beta^2 z^2 - (\alpha + \beta z)^2) - 2z^2(\alpha + \beta z)^2] = 0.$$

Finally,

$$\varphi_\nu(z) = \frac{\alpha\beta z(1 + z)^2 + (\alpha - \beta)^2 z^2}{(1 - z^2)^2}$$

as desired. □

The next proposition provides a Lebesgue decomposition of the spectral measure ν_∞ .

Proposition 2.6. *One has*

$$\nu_\infty = a\delta_\pi + b\delta_0 + \frac{\sqrt{-(\cos \phi - r_+)(\cos \phi - r_-)}}{2\pi|\sin \phi|} \mathbf{1}_{(\phi_-, \phi_+) \cup (-\phi_+, -\phi_-)} d\phi$$

with

$$a = \frac{|\alpha - \beta|}{2}, b = \frac{|\alpha + \beta|}{2}, r_\pm = \alpha\beta \pm \sqrt{(1 - \alpha^2)(1 - \beta^2)} \quad \text{and} \quad \phi_\pm = \arccos r_\pm.$$

Proof. Writing (2.10) as

$$H(\infty, z) = \frac{\sqrt{(1 - z^2)^2 + 4z[\alpha\beta(1 + z)^2 + (\alpha - \beta)^2 z]}}{(1 - z^2)},$$

it follows that $H(\infty, \cdot)$ admits two simple poles at $z = 1$ and $z = -1$. So that, the decomposition of ν_∞ is given by

$$\nu_\infty = a\delta_\pi + b\delta_0 + \Re [H(\infty, e^{i\phi})] \frac{d\phi}{2\pi}$$

where $d\phi$ denotes the (no-normalized) Lebesgue measure on $\mathbb{T} = (-\pi, \pi]$ and a, b are the residue of $\frac{1}{2}H(\infty, \cdot)$ at $-1, 1$. Thus, we have

$$a = \lim_{z \rightarrow -1} \frac{\sqrt{(1 - z^2)^2 + 4z[\alpha\beta(1 + z)^2 + (\alpha - \beta)^2 z]}}{2(1 + z)} = \frac{|\alpha - \beta|}{2},$$

$$b = \lim_{z \rightarrow 1} \frac{\sqrt{(1 - z^2)^2 + 4z[\alpha\beta(1 + z)^2 + (\alpha - \beta)^2 z]}}{2(1 - z)} = \frac{|\alpha + \beta|}{2}$$

and the density is given by direct calculation

$$\begin{aligned}
\Re [H(\infty, e^{i\phi})] &= \Re \left[\sqrt{1 + 4e^{i\phi} \frac{\alpha\beta(1 + e^{i\phi})^2 + (\alpha - \beta)^2 e^{i\phi}}{(1 - e^{2i\phi})^2}} \right] \\
&= \Re \left[\sqrt{1 + \frac{4\alpha\beta e^{i\phi}}{(1 - e^{i\phi})^2} + \frac{4(\alpha - \beta)^2 e^{2i\phi}}{(1 - e^{2i\phi})^2}} \right] \\
&= \sqrt{1 - \frac{\alpha\beta}{\sin^2 \frac{\phi}{2}} - \frac{(\alpha - \beta)^2}{\sin^2 \phi}} \\
&= \frac{\sqrt{\sin^2 \phi - 4\alpha\beta \cos^2 \frac{\phi}{2} - (\alpha - \beta)^2}}{|\sin \phi|},
\end{aligned}$$

where we have used in the last equality the relation

$$\sin^2 \frac{\phi}{2} = \frac{\sin^2 \phi}{4 \cos^2 \frac{\phi}{2}}.$$

Finally, by use of the basic trigonometric identities:

$$\cos^2 \phi + \sin^2 \phi = 1 \quad \text{and} \quad \cos^2 \frac{\phi}{2} = \frac{1 + \cos \phi}{2},$$

the denominator rewrites as

$$\begin{aligned}
\sin^2 \phi - 4\alpha\beta \cos^2 \frac{\phi}{2} - (\alpha - \beta)^2 &= 1 - \cos^2 \phi - 2\alpha\beta \cos \phi - 2\alpha\beta - (\alpha - \beta)^2 \\
&= -\cos^2 \phi - 2\alpha\beta \cos \phi + 1 - \alpha^2 - \beta^2.
\end{aligned}$$

Using the discriminant $\Delta = 4(\alpha^2\beta^2 + 1 - \alpha^2 - \beta^2) = 4(1 - \alpha^2)(1 - \beta^2) \geq 0$, we get the factorization $-(\cos \phi - r_+)(\cos \phi - r_-)$ with

$$r_{\pm} = \alpha\beta \pm \sqrt{(1 - \alpha^2)(1 - \beta^2)}.$$

□

Remark 2.7. *It should be noted that this measure appears in [9, Example 4.5] as the distribution of $e^{i\pi P} e^{-i\pi Q}$ for a pair of free projections $\{P, Q\}$ in \mathcal{A} . In particular, when $\alpha = \beta = 0$ (i.e. $\tau(P) = \tau(Q) = 1/2$), it coincides with the uniform measure on \mathbb{T} .*

3. SUBORDINATION RESULTS

The aim of this section is to derive an exact subordination relation in terms of Löwner equations and give an explicit formula for its unique subordinate family.

Proposition 3.1. *Let H be a solution to the PDE (2.9). Then there exists a unique subordinate family of conformal self-maps v_t on \mathbb{D} such that*

$$H(t, v_t(z))^2 - H(\infty, v_t(z))^2 = H(0, z)^2 - H(\infty, z)^2. \quad (3.1)$$

Proof. Differentiating the characteristic curve $t \mapsto (v_t(z), H(t, v_t(z)))$ associated with the PDE (2.9), we get the following system of ODEs:

$$\partial_t v_t = v_t H(t, v_t), \quad v_0(z) = z, \quad (3.2)$$

$$\partial_t [H(t, v_t)] = \frac{4(\alpha^2 + \beta^2)v_t^2(1 + v_t^2) + 2\alpha\beta v_t(1 + 6v_t^2 + v_t^4)}{(1 - v_t^2)^3}. \quad (3.3)$$

The ODE (3.2) is the radial Löwner equation driven by the Herglotz function H . Then v_t is a conformal map from $\mathbb{D}_t := \{z, T_z > t\}$ onto \mathbb{D} (see, e.g., Theorem 4.14 in [12]), where T_z is the supremum of all t such that $v_t(z) \in \mathbb{D}$ for fixed $z \in \mathbb{D}$. The ODE (3.3), combined with (3.2), shows that

$$H\partial_t H = \frac{4(\alpha^2 + \beta^2)v_t(1 + v_t^2) + 2\alpha\beta(1 + 6v_t^2 + v_t^4)}{(1 - v_t^2)^3} \partial_t v_t. \quad (3.4)$$

Which implies, after integrating with respect to t , that

$$\begin{aligned} H(t, v_t(z))^2 - H(0, z)^2 &= 4 \frac{(\alpha^2 + \beta^2)v_t(z)^2 + 2\alpha\beta v_t(z)(1 + v_t(z)^2)}{(1 - v_t(z)^2)^2} \\ &\quad - 4 \frac{(\alpha^2 + \beta^2)z^2 + \alpha\beta z(1 + z^2)}{(1 - z^2)^2}. \end{aligned}$$

This proves the proposition. \square

Remark 3.2. When P, Q are two projections associated to R, S such that $\tau(P) = \tau(Q) = 1/2$ (i.e. $\alpha = \beta = 0$), the function $t \mapsto H(t, v_t(z))$ is constant, so that $H(t, v_t(z)) = H(0, z)$. Then, $v_t(z) = ze^{tH(0, z)}$. This enables us to retrieve the description of $\nu_{t/2}$ in [11, Proposition 3.3]. In particular, when $P = Q$ and $\nu_0 = \delta_0$ (i.e. $H(0, z) = (1 + z)/(1 - z)$), we retrieve the description in [5, Corollary 3.3] of the spectral measure μ_t on $[0, 1]$ of the free Jacobi process (the process X_t viewed as a random variable in the compressed probability space $(P\mathcal{A}P, \frac{1}{\tau(P)}\tau)$).

The next proposition gives an explicit expression for the subordinate family $(v_t)_{t \geq 0}$.

Proposition 3.3. For any $t \geq 0$ and $z \in \mathbb{D}_t$, we have

$$v_t(z) = \frac{w_t(y) - 1}{w_t(y) + 1},$$

with

$$w_t(y) = \sqrt{\frac{(b^2 - a^2 - c + de^{t\sqrt{c}})^2 - 4a^2c}{(b^2 - a^2 + c + de^{t\sqrt{c}})^2 - 4b^2c}}, \quad y = \frac{1 + z}{1 - z},$$

where $a = \frac{|\alpha - \beta|}{2}$, $b = \frac{|\alpha + \beta|}{2}$,

$$c = c(y) := \max\{\alpha^2, \beta^2\} + F(0, y)^2 - \left(\frac{a + by^2}{y}\right)^2$$

and

$$d = d(y) := -c - \alpha\beta + \frac{2c - 2\sqrt{c}\sqrt{c - (c + \alpha\beta)(1 - y^2) + b^2(1 - y^2)^2}}{1 - y^2}.$$

Proof. In order to make easier computations, we use the Möbius transform

$$z \mapsto y = \frac{1 + z}{1 - z}$$

to introduce The function $F(t, y) := H(t, z)$. Since $\frac{dy}{dz} = \frac{-2z}{(1-z)^2}$, the PDE (2.9) becomes

$$\partial_t F + \frac{y^2 - 1}{4} \partial_y F^2 = \frac{(y^2 - 1)}{8y^3} ((\alpha + \beta)^2 y^4 - (\alpha - \beta)^2). \quad (3.5)$$

As usual, the characteristic curve $t \mapsto (w_t(z), F(t, w_t(z)))$ associated with the PDE (3.5) satisfies the system of ODEs:

$$\partial_t w_t = \frac{1}{2}(w_t^2 - 1)F(t, w_t), \quad w_0(y) = y, \quad (3.6)$$

$$\partial_t [F(t, w_t)] = \frac{(w_t^2 - 1)}{8w_t^3} ((\alpha + \beta)^2 w_t^4 - (\alpha - \beta)^2), \quad (3.7)$$

with

$$w_t(y) := \frac{1 + v_t(z)}{1 - v_t(z)}.$$

Combining the two last ODE's, we get

$$F \partial_t F = \frac{(\alpha + \beta)^2 w_t^4 - (\alpha - \beta)^2}{4w_t^3}.$$

Hence, integrating with respect to t , we get

$$\begin{aligned} F(t, w_t(y))^2 &= F(0, y^2) + \frac{(\alpha + \beta)^2 w_t^4(y) + (\alpha - \beta)^2}{4w_t^2(y)} - \frac{(\alpha + \beta)^2 y^4 + (\alpha - \beta)^2}{4y^2} \\ &= 1 + F^2(0, y) - F^2(\infty, y) - \frac{\alpha^2 + \beta^2}{2} + \frac{(\alpha + \beta)^2 w_t^4(y) + (\alpha - \beta)^2}{4w_t^2(y)}. \end{aligned}$$

So that, the ODE (3.6) becomes

$$\partial_t w_t(y) = \frac{w_t^2(y) - 1}{2} \sqrt{1 + F^2(0, y) - F^2(\infty, y) - \frac{\alpha^2 + \beta^2}{2} + \frac{(\alpha + \beta)^2 w_t^4(y) + (\alpha - \beta)^2}{4w_t^2(y)}}.$$

Or, equivalently

$$\partial_t w_t(y) = \frac{w_t^2(y) - 1}{2w_t(y)} \sqrt{b^2 w_t^4(y) + [1 + F^2(0, y) - F^2(\infty, y) - a^2 - b^2] w_t(y)^2 + a^2}.$$

In order to solve this last ODE, we are lead to compute the indefinite integral for $y > 0$

$$-2 \int \frac{xdx}{(1 - x^2) \sqrt{b^2 x^4 + (1 + F(0, y)^2 - F(\infty, y)^2 - a^2 - b^2) x^2 + a^2}}.$$

Performing the variable change $u = 1 - x^2$, we transform this integral to

$$\int \frac{du}{u \sqrt{c - c_1 u + c_2 u^2}}$$

with

$$\begin{aligned} c &= 1 + F(0, y)^2 - F(\infty, y)^2, \\ c_1 &= c + b^2 - a^2 = c + \alpha\beta, \\ c_2 &= b^2 = \frac{(\alpha + \beta)^2}{4}. \end{aligned}$$

Then writing

$$\begin{aligned}
c &= F(0, y)^2 - \frac{(\alpha + \beta)^2 y^4 + (\alpha - \beta)^2}{4y^2} + \frac{\alpha^2 + \beta^2}{2} \\
&= F(0, y)^2 - \left(\frac{|\alpha + \beta|y^2 + |\alpha - \beta|}{2y} \right)^2 + \frac{\alpha^2 + \beta^2 + |\alpha^2 - \beta^2|}{2} \\
&= F(0, y)^2 - \left(\frac{by^2 + a}{y} \right)^2 + \max\{\alpha^2, \beta^2\},
\end{aligned}$$

we get

$$c_1^2 - 4cc_2 = c^2 + 2c\alpha\beta + (\alpha\beta)^2 - c(\alpha + \beta)^2 = (c - \alpha^2)(c - \beta^2).$$

Hence (see the proof in [6, Theorem 3]), we have

$$\int \frac{du}{u\sqrt{c - c_1u + c_2u^2}} = \frac{1}{\sqrt{c}} \ln \frac{2c - c_1u - 2\sqrt{c}\sqrt{c - c_1u + c_2u^2}}{|u|}.$$

Let $u_t(y) := 1 - w_t^2(y)$, then

$$\frac{2c - c_1u_t(y) - 2\sqrt{c}\sqrt{c - c_1u_t(y) + c_2u_t(y)^2}}{|u_t(y)|} = de^{t\sqrt{c}}$$

for some $d = d(y, \alpha, \beta)$ and hence

$$2c - (c_1 + \epsilon de^{t\sqrt{c}})u_t(y) = 2\sqrt{c}\sqrt{c - c_1u_t(y) + c_2u_t(y)^2}$$

where ϵ is the sign of u . Raising this equality to the square and rearranging it, we get

$$\left[(c_1 + \epsilon de^{t\sqrt{c}})^2 - 4cc_2 \right] u_t(y) = 4c\epsilon de^{t\sqrt{c}}. \quad (3.8)$$

Equivalently,

$$u_t(y) = \frac{4c\tilde{d}e^{t\sqrt{c}}}{(c_1 + \tilde{d}e^{t\sqrt{c}})^2 - 4cc_2}$$

with $\tilde{d} = \epsilon d$. Hence

$$\begin{aligned}
w_t(y)^2 &= \frac{(c_1 + \tilde{d}e^{t\sqrt{c}})^2 - 4cc_2 - 4c\tilde{d}e^{t\sqrt{c}}}{(c_1 + \tilde{d}e^{t\sqrt{c}})^2 - 4cc_2} \\
&= \frac{(b^2 - a^2 + c + \tilde{d}e^{t\sqrt{c}})^2 - 4cb^2 - 4c\tilde{d}e^{t\sqrt{c}}}{(b^2 - a^2 + c + \tilde{d}e^{t\sqrt{c}})^2 - 4cb^2} \\
&= \frac{(b^2 - a^2 - c + \tilde{d}e^{t\sqrt{c}})^2 - 4ca^2}{(b^2 - a^2 + c + \tilde{d}e^{t\sqrt{c}})^2 - 4cb^2}. \quad (3.9)
\end{aligned}$$

Finally, in order to find the value of \tilde{d} , we check the equality (3.8) for $t = 0$

$$\left[(c_1 + \tilde{d})^2 - 4cc_2 \right] u_0 = 4c\tilde{d}$$

where $u_0 := u_0(y) = 1 - w_0(y)^2 = 1 - y^2$. Then

$$\tilde{d}^2 + 2\left(c_1 - \frac{2c}{u_0}\right)\tilde{d} + c_1^2 - 4cc_2 = 0.$$

The discriminant of this quadratic is

$$\begin{aligned}\Delta' &= \left(c_1 - \frac{2c}{u_0}\right)^2 - c_1^2 + 4cc_2 \\ &= \frac{4c^2}{u_0^2} - \frac{4cc_1}{u_0} + 4cc_2 \\ &= \frac{4c}{u_0^2} (c - c_1u_0 + c_2u_0^2)\end{aligned}$$

and hence

$$\tilde{d} = -c_1 + \frac{2c}{u_0} \pm \frac{2\sqrt{c}}{u_0} \sqrt{c - c_1u_0 + c_2u_0^2}.$$

When $a = b = 0$ (i.e. $c_1 = c = F(0, y)^2$ and $c_2 = 0$), it becomes

$$\tilde{d} = -c + \frac{2c \pm 2cy}{1 - y^2} = \frac{1 \pm 2y + y^2}{1 - y^2} c.$$

Therefore the only solution is

$$\tilde{d} = -c_1 + \frac{2c}{u_0} - \frac{2\sqrt{c}}{u_0} \sqrt{c - c_1u_0 + c_2u_0^2}$$

since for $a = b = 0$, we have on the one hand by (3.9)

$$w_t(y) = \sqrt{\frac{(-c + \tilde{d}e^{t\sqrt{c}})^2}{(c + \tilde{d}e^{t\sqrt{c}})^2}} = \frac{1 - \frac{\tilde{d}}{c}e^{t\sqrt{c}}}{1 + \frac{\tilde{d}}{c}e^{t\sqrt{c}}}$$

on the other hand (see Remark (3.2)),

$$w_t(y) = \frac{1 + v_t(z)}{1 - v_t(z)} = \frac{1 + ze^{tH(0,z)}}{1 - ze^{tH(0,z)}} = \frac{1 + \frac{y-1}{y+1}e^{tF(0,y)}}{1 - \frac{y-1}{y+1}e^{tF(0,y)}}.$$

Hence we are done. \square

Remark 3.4. Note that (see, e.g., [12, Remark 4.15]) the inverse $\kappa_t := v_t^{-1} : \mathbb{D} \mapsto \mathbb{D}_t$ satisfies

$$\partial_t \kappa_t(z) = -z \partial_z \kappa_t(z) H(t, z), \quad \kappa_0(z) = z,$$

the radial Löwner PDE driven by the probability measure ν_t .

Define¹

$$K(t, z) := \sqrt{\frac{1}{4}H(t, z)^2 - \frac{1}{4} \left(a \frac{1-z}{1+z} + b \frac{1+z}{1-z} \right)^2}, \quad |z| < 1. \quad (3.10)$$

This function is analytic in \mathbb{D} with positive real part. Indeed, the function

$$H(t, z)^2 - \left(a \frac{1-z}{1+z} + b \frac{1+z}{1-z} \right)^2, \quad |z| < 1$$

¹We take the principal branch of the square root.

can not take negative value in \mathbb{D} since the two measures $\nu_t - a\delta_\pi - b\delta_0$ and $\nu_t + a\delta_\pi + b\delta_0$ are finite positive measure in \mathbb{T} (see Proposition 4.5 below). Thus, according to the Herglotz theorem, there exists a unique probability measure γ_t in \mathbb{T} such that

$$K(t, z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\gamma_t(w). \quad (3.11)$$

Remark 3.5. Observe that from PDE (2.9), the function K satisfies

$$\partial_t K + zH\partial_z K = 0.$$

It becomes, in the time stationary case, the constant $\sqrt{1 - \max\{\alpha^2, \beta^2\}}/2$.

Proposition 3.6. The exact subordination $K(t, z) = K(0, \kappa_t(z))$ holds for any $z \in \mathbb{D}$.

Proof. From (3.1), we have

$$4K(t, v_t(z))^2 = H(0, z)^2 - H(\infty, z)^2 + H(\infty, v_t(z))^2 - \left(a \frac{1 - v_t(z)}{1 + v_t(z)} + b \frac{1 + v_t(z)}{1 - v_t(z)} \right)^2.$$

But

$$\begin{aligned} H(\infty, z)^2 &= 1 + 4z \frac{\alpha\beta(1+z)^2 + (\alpha - \beta)^2 z}{(1 - z^2)^2} \\ &= 1 - \max\{\alpha^2, \beta^2\} + \left(a \frac{1 - z}{1 + z} + b \frac{1 + z}{1 - z} \right)^2. \end{aligned}$$

Then

$$4K(t, v_t(z))^2 = H(0, z)^2 - \left(a \frac{1 - z}{1 + z} + b \frac{1 + z}{1 - z} \right)^2,$$

and we are done. \square

Lemma 3.7. The region $\overline{\mathbb{D}_t}$ does not contain 1 (resp. -1) whenever $b > 0$ or $b = 0$ and $\nu_0\{0\} > 0$ (resp. $a > 0$ or $a = 0$ and $\nu_0\{\pi\} > 0$).

Proof. By contradiction, assume that $1 \in \overline{\mathbb{D}_t}$. Then, since the restriction of v_t to real numbers is real and since v_t is an increasing function on $(0,1)$, $\lim_{r \rightarrow 1^-} v_t(r) = 1$. Equivalently, in the y -variable we have, $\lim_{y \rightarrow +\infty} w_t(y) = \infty$ (see Proposition 3.3). But since $\nu_0\{0\} \geq b$ (see Proposition 4.5 below), the assumption $b > 0$ or $b = 0$ and $\nu_0\{0\} > 0$ implies

$$\lim_{y \rightarrow +\infty} c(y) = \lim_{y \rightarrow +\infty} \left[F(0, y) - by - \frac{a}{y} \right] \left[F(0, y) + by + \frac{a}{y} \right] + \max\{\alpha^2, \beta^2\} = +\infty,$$

and

$$\lim_{y \rightarrow +\infty} d(y) = \lim_{y \rightarrow +\infty} c \left[-1 - \frac{\alpha\beta}{c} + \frac{2}{1 - y^2} + 2\sqrt{\frac{1}{(y^2 - 1)^2} + \frac{c + \alpha\beta}{c(y^2 - 1)} + \frac{b^2}{c}} \right] = -\infty.$$

So that, $\lim_{y \rightarrow +\infty} c(y)/d(y) = -1$. Thus, $\lim_{y \rightarrow +\infty} w_t^2(y) = 1$ which is absurd. Proceeding in the same way, we prove that $\overline{\mathbb{D}_t}$ does not contain -1 since v_t is a decreasing function on $(-1,0)$ and since we have the limits $\lim_{y \rightarrow 0} c(y) = +\infty$, $\lim_{y \rightarrow 0} c(y)/d(y) = 1$. \square

Corollary 3.8. Assume that $H(0, \kappa_t(z))$ is a function of Hardy class $H^3(\mathbb{D})$. Then $K(t, \cdot)$ is a function of Hardy class $H^3(\mathbb{D})$ for any $t > 0$.

Proof. For simplicity, we assume that both $a > 0$ and $b > 0$. Since the boundary of \mathbb{D}_t is at a positive distance from the points ± 1 (see Lemma 3.7), one easily sees that

$$z \mapsto \left(a \frac{1 - \kappa_t(z)}{1 + \kappa_t(z)} + b \frac{1 + \kappa_t(z)}{1 - \kappa_t(z)} \right)^2$$

is a function of Hardy class $H^\infty(\mathbb{D})$ for any $t > 0$. Moreover, the assumption in this corollary implies that $H(0, \kappa_t(z))^2 \in H^{3/2}(\mathbb{D})$. By the subordination relation

$$2K(t, z) = \sqrt{H(0, \kappa_t(z))^2 - \left(a \frac{1 - \kappa_t(z)}{1 + \kappa_t(z)} + b \frac{1 + \kappa_t(z)}{1 - \kappa_t(z)} \right)^2},$$

$K(t, z)$ becomes therefore a function of Hardy class $H^3(\mathbb{D})$ for any $t > 0$. □

4. RELATIONSHIP BETWEEN μ_t AND ν_t

Keep the symbols $P, Q, R, S, \alpha, \beta, a, b$ and μ_t, ν_t above. In what follows P, Q and R, S are associated. Our goal here is to derive relationship between μ_t and ν_t and give more detailed properties of ν_t . The following proposition gives a relationship between the corresponding sequence of moments.

Proposition 4.1. *For any $n \geq 1$, one has :*

$$\tau[(PU_tQU_t^*)^n] = \frac{1}{2^{2n+1}} \binom{2n}{n} + \frac{\tau(R+S)}{4} + \frac{1}{2^{2n}} \sum_{k=1}^n \binom{2n}{n-k} \tau((RU_tSU_t^*)^k). \quad (4.1)$$

Proof. We write

$$\tau[(PU_tQU_t^*)^n] = \frac{1}{2^{2n}} \tau[(\mathbf{1} + R)U_t(\mathbf{1} + S)U_t^*]^n].$$

Let $\tilde{S} := U_tSU_t^*$. Then writing

$$(\mathbf{1} + R)U_t(\mathbf{1} + S)U_t^* = (\mathbf{1} + R)(\mathbf{1} + \tilde{S}).$$

one easily can see that the same enumeration techniques used in [5, Proposition 4.1] to expand $\tau[(\mathbf{1} + R)(\mathbf{1} + \tilde{S})^n]$ remain valid, but here we will take into account contribution of words formed by an odd number of letters. Using the trace property and the relations $R^2 = \tilde{S}^2 = \mathbf{1}$, this contribution is $\tau(R) + \tau(S)$ up to a positive integer N . By letting $R = S$ and using the expansion in [5, p 1366], we get $2N = 2^{2n-1}$ and hence the desired equality follows. □

Let

$$G(t, z) := \frac{1}{z} + \sum_{n \geq 1} \frac{\tau[(PU_tQU_t^*)^n]}{z^{n+1}}, \quad t \geq 0, |z| > 1,$$

be the Cauchy transform of the process X_t . The following corollary gives a relationship between G and the Herglotz transform of ν_t .

Corollary 4.2. *One has*

$$G(t, z) = \frac{1}{2z} + \frac{\alpha + \beta}{4z(z-1)} + \frac{H(t, g(z))}{2\sqrt{z^2 - z}}, \quad t \geq 0, |z| > 1, \quad (4.2)$$

where²

$$g(z) = 2z - 1 + 2\sqrt{z^2 - z}.$$

Proof. We will prove the following equivalent relation

$$\psi_{\mu_t}(z) = \frac{(\alpha + \beta + 2)z - 2}{4(1 - z)} + \frac{H(t, g(1/z))}{2\sqrt{1 - z}}, \quad t \geq 0, |z| < 1,$$

satisfied by the moment generating function of the process X_t

$$\psi_{\mu_t}(z) := \sum_{n \geq 1} \tau[(PU_t QU_t^*)^n] z^n, \quad t \geq 0, |z| < 1.$$

Before going into the details, recall from [5] that $|g(1/z)| \leq |z| < 1$ in the open unit disc, then this last relation makes sense for all $|z| < 1$. Now multiplying (4.1) by z^n and summing over $n \geq 1$, we get

$$\psi_{\mu_t}(z) = \frac{1}{2\sqrt{1 - z}} - \frac{1}{2} + \frac{(\alpha + \beta)z}{4(1 - z)} + \sum_{n \geq 1} \frac{z^n}{2^{2n}} \sum_{k=1}^n \binom{2n}{n-k} \tau[(RU_t SU_t^*)^k].$$

But, this last term rewrites, after permutation of sums and reindexing $j = n - k$, as

$$\sum_{n \geq 1} \frac{z^n}{2^{2n}} \sum_{k=1}^n \binom{2n}{n-k} \tau[(RU_t SU_t^*)^k] = \sum_{k \geq 1} \tau[(RU_t SU_t^*)^k] \sum_{j \geq 0} \frac{z^{j+k}}{2^{2j+2k}} \binom{2j+2k}{j}.$$

Using the identity (see, e.g. [5])

$$\sum_{j \geq 0} \binom{2j+2k}{j} \frac{z^j}{2^{2j}} = \frac{2^{2k}}{\sqrt{1 - z}} (1 + \sqrt{1 - z})^{-2k}, \quad |z| < 1,$$

we get

$$\begin{aligned} \psi_{\mu_t}(z) &= \frac{1}{2\sqrt{1 - z}} - \frac{1}{2} + \frac{(\alpha + \beta)z}{4(1 - z)} + \frac{1}{\sqrt{1 - z}} \sum_{k \geq 1} \frac{\tau[(RU_t SU_t^*)^k] z^k}{(1 + \sqrt{1 - z})^{2k}} \\ &= \frac{1}{2\sqrt{1 - z}} - \frac{1}{2} + \frac{(\alpha + \beta)z}{4(1 - z)} + \frac{1}{\sqrt{1 - z}} \frac{H(t, g(1/z)) - 1}{2} \\ &= -\frac{1}{2} + \frac{(\alpha + \beta)z}{4(1 - z)} + \frac{H(t, g(1/z))}{2\sqrt{1 - z}}, \end{aligned}$$

which proves the corollary. □

We are now ready to prove the relationship $\mu_t \longleftrightarrow \nu_t$ between the spectral measure of X_t and Y_t .

²The principal branch of the square root is taken.

Theorem 4.3. Let $\tilde{\mu}_t(d\theta)$ be the positive measure on $[0, \pi]$ obtained from $\mu_t(dx)$ via the variable change $x = \cos^2(\theta/2)$ and $\hat{\mu}_t := \frac{1}{2}(\tilde{\mu}_t + (\tilde{\mu}_t|_{(0,\pi)}) \circ j^{-1})$ its symmetrization on $(-\pi, \pi)$ with the mapping $j : \theta \in (0, \pi) \mapsto -\theta \in (-\pi, 0)$. Then, the two measures μ_t and ν_t are related via

$$\nu_t = 2\hat{\mu}_t - \frac{2 - \alpha - \beta}{2}\delta_\pi - \frac{\alpha + \beta}{2}\delta_0. \quad (4.3)$$

Proof. By (4.2), we have

$$H(t, g(z)) = 2\sqrt{z^2 - z} \left(G(t, z) - \frac{2 - \alpha - \beta}{4z} - \frac{\alpha + \beta}{4(z - 1)} \right).$$

Letting $\tilde{\mu}_t(d\theta) = \mu_t(dx)$ with $x = \cos^2(\theta/2)$, $\theta \in [0, \pi]$, we get

$$H(t, g(z)) = -2\sqrt{z^2 - z} \left(\int_0^\pi \frac{1}{z - \cos^2 \frac{\theta}{2}} \tilde{\mu}_t(d\theta) - \frac{2 - \alpha - \beta}{4z} - \frac{\alpha + \beta}{4(z - 1)} \right).$$

Next, we perform the variable change

$$\zeta := g(z) = 2z - 1 + 2\sqrt{z^2 - z} \Leftrightarrow z = \frac{2 + \zeta + \zeta^{-1}}{4},$$

to get

$$\begin{aligned} H(t, \zeta) &= \frac{\zeta^{-1} - \zeta}{2} \left(\int_0^\pi \frac{1}{\frac{2 + \zeta + \zeta^{-1}}{4} - \cos^2 \frac{\theta}{2}} \tilde{\mu}_t(d\theta) - \frac{2 - \alpha - \beta}{2 + \zeta + \zeta^{-1}} - \frac{\alpha + \beta}{-2 + \zeta + \zeta^{-1}} \right) \\ &= \int_0^\pi \frac{2(\zeta^{-1} - \zeta)}{2 + \zeta + \zeta^{-1} - 4 \cos^2 \frac{\theta}{2}} \tilde{\mu}_t(d\theta) - \frac{(2 - \alpha - \beta)(1 - \zeta)}{2(1 + \zeta)} - \frac{(\alpha + \beta)(1 + \zeta)}{2(1 - \zeta)}. \end{aligned}$$

But since

$$\begin{aligned} \frac{\zeta^{-1} - \zeta}{2 + \zeta + \zeta^{-1} - 4 \cos^2 \frac{\theta}{2}} &= \frac{\zeta^{-1} - \zeta}{\zeta + \zeta^{-1} - 2 \cos \theta} \\ &= \frac{1 - \zeta^2}{\zeta^2 - 2\zeta \cos \theta + 1} \\ &= \frac{e^{i\theta}}{e^{i\theta} - \zeta} + \frac{e^{i\theta}}{e^{-i\theta} - \zeta} - 1, \end{aligned}$$

then

$$H(t, \zeta) = 2 \int_0^\pi \left(\frac{e^{i\theta}}{e^{i\theta} - \zeta} + \frac{e^{-i\theta}}{e^{-i\theta} - \zeta} - 1 \right) \tilde{\mu}_t(d\theta) - \frac{(2 - \alpha - \beta)(1 - \zeta)}{2(1 + \zeta)} - \frac{(\alpha + \beta)(1 + \zeta)}{2(1 - \zeta)}.$$

Thus, using the symmetrization $\hat{\mu}_t := \frac{1}{2}(\tilde{\mu}_t + (\tilde{\mu}_t|_{(0,\pi)}) \circ j^{-1})$ with $j : \theta \in (0, \pi) \mapsto -\theta \in (-\pi, 0)$, we get

$$H(t, \zeta) = \int_{-\pi}^\pi \frac{e^{i\theta} + \zeta}{e^{i\theta} - \zeta} \left(2\hat{\mu}_t - \frac{2 - \alpha - \beta}{2}\delta_\pi - \frac{\alpha + \beta}{2}\delta_0 \right)(d\theta).$$

This proves the theorem. □

Remark 4.4. The relationship $\mu_t \rightsquigarrow \nu_t$ enable us, in particular, to retrieve the decomposition of ν_∞ already obtained in section 2 from the spectral measure μ_∞ (given by the free multiplicative convolution of the spectral measure of P and UQU^* with U is a Haar unitary free from $\{P, Q\}$ (see, [8, Example 3.6.7])). Indeed, we have $\hat{\delta}_0 = \delta_\pi, \hat{\delta}_1 = \delta_0$ and if μ_t has the density $h(x)$ with respect to dx on $[0, 1]$, then ν_t has the density $\hat{h}(\phi)$ with respect to the (no-normalized) Lebesgue measure $d\phi$ on $\mathbb{T} = (-\pi, \pi]$ with $\hat{h}(\phi) = h(\cos^2(\phi/2))|\sin \phi|/4$.

By virtue of the fact that P and $U_tQU_t^*$ are in generic position for any $t > 0$ (see, e.g., [11, Remark 3.5]), we have

Proposition 4.5. The positive measure $\sigma_t := \nu_t - a\delta_\pi - b\delta_0$ with $a = \frac{|\alpha-\beta|}{2}$ and $b = \frac{|\alpha+\beta|}{2}$, has no atom at both 0 and π for every $t > 0$. Moreover, at $t = 0$, $\sigma_0\{0\} \geq 0, \sigma_0\{\pi\} \geq 0$ with equalities (i.e. σ_0 has no atom at both 0 and π), if and only if the projections P and Q are in generic position.

Proof. By (4.3), we have

$$\begin{aligned}\sigma_t &= 2\hat{\mu}_t - \frac{2 - \alpha - \beta + |\alpha - \beta|}{2}\delta_\pi - \frac{\alpha + \beta + |\alpha + \beta|}{2}\delta_0 \\ &= 2\hat{\mu}_t - (1 - \min\{\alpha, \beta\})\delta_\pi - \max\{\alpha + \beta, 0\}\delta_0.\end{aligned}$$

Since $\alpha = 2\tau(P) - 1$ and $\beta = 2\tau(Q) - 1$, then we write

$$\sigma_t = 2[\hat{\mu}_t - (1 - \min\{\tau(P), \tau(Q)\})\delta_\pi - \max\{\tau(P) + \tau(Q) - 1, 0\}\delta_0],$$

and hence the desired assertions follows immediately from [11, Proposition 3.1]. \square

Remark 4.6. When $\alpha = \beta = 0$ (i.e. $\tau(P) = \tau(Q) = 1/2$), the two measures $\sigma_{t/2}$ and $\gamma_{t/2}$ (recall the definition of γ_t from (3.11)) coincide with the spectral measure of the product of the free unitary BM with a free unitary operator whose distribution is $\sigma_0 = 2\hat{\mu}_0$.

5. FREE MUTUAL INFORMATION AND ORBITAL FREE ENTROPY

Here is our main application to the proof of the conjecture $i^* = -\chi_{orb}$, for any pair of projections. For a given pair of projections P, Q , we use the same definitions of the free mutual information $i^*(P : Q)$ and the orbital free entropy $\chi_{orb}(P, Q)$ as expounded in the last section of the paper [11]. The following equality was obtained in [11] (note that the function H there is exactly our K^2).

Lemma 5.1. [11, Lemma 4.4] If $K(t, \cdot)$ define a function of Hardy class $H^3(\mathbb{D})$ for any $t > 0$, then

$$i^*(\mathbb{C}P + \mathbb{C}(I - P); \mathbb{C}Q + \mathbb{C}(I - Q)) = -\chi_{orb}(P, Q)$$

holds.

As a consequence of Corollary 3.8 together with this Lemma, we have the following.

Theorem 5.2. Under the same assumption as in Corollary 3.8, the equality

$$i^*(\mathbb{C}P + \mathbb{C}(I - P); \mathbb{C}Q + \mathbb{C}(I - Q)) = -\chi_{orb}(P, Q)$$

holds.

In particular,

Corollary 5.3. Assume that σ_0 (see Proposition (4.5)) has an L^3 -density with respect to $d\theta$. Then

$$i^*(\mathbb{C}P + \mathbb{C}(I - P); \mathbb{C}Q + \mathbb{C}(I - Q)) = -\chi_{orb}(P, Q)$$

holds.

Proof. Define

$$\tilde{H}(t, z) := \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\sigma_t(\theta).$$

Under the assumption here and according to [3, Theorem 1.7, p.208], $\tilde{H}(0, z)$ is a function of Hardy class $H^3(\mathbb{D})$ and hence so does $\tilde{H}(0, \kappa_t(z))$ too by Littlewood's subordination theorem (see [7, Theorem 1.7]). Moreover, since the function

$$a \frac{1 - \kappa_t(z)}{1 + \kappa_t(z)} + b \frac{1 + \kappa_t(z)}{1 - \kappa_t(z)}$$

is of hardy class $H^\infty(\mathbb{D})$, then one easily can see that the function

$$H(0, \kappa_t(z)) = \tilde{H}(0, \kappa_t(z)) + a \frac{1 - \kappa_t(z)}{1 + \kappa_t(z)} + b \frac{1 + \kappa_t(z)}{1 - \kappa_t(z)}$$

is of hardy class $H^3(\mathbb{D})$ and hence we are done by using Corollary 3.8. \square

Remark 5.4. From the relationship $\mu_t \rightsquigarrow \nu_t$ and using the equality

$$\sigma_0 = 2[\hat{\mu}_0 - (1 - \min\{\tau(P), \tau(Q)\})\delta_\pi - \max\{\tau(P) + \tau(Q) - 1, 0\}\delta_0],$$

one can see that the assumption of the above Corollary is equivalent to

$$\mu_0 - (1 - \min\{\tau(P), \tau(Q)\})\delta_0 - \max\{\tau(P) + \tau(Q) - 1, 0\}\delta_1$$

has an L^3 -density with respect to $x(1 - x)dx$ on $[0, 1]$. This gives an improvement to the result in [11, Corollary 4.5].

Remark 5.5. Finally, we should probably point out that the assumption of Corollary 3.8 suggests a more detailed study of the boundary of \mathbb{D}_t . With regard to this question, we might note that Proposition 3.3 shows that

$$\partial\mathbb{D}_t = \left\{ z \in \overline{\mathbb{D}} : w_t(y)^2 \in [1, \infty), y = \frac{1 + z}{1 - z} \right\}.$$

The recent preprint [4], provides an explicit equation and more detailed properties of the boundary of \mathbb{D}_t when the two projections coincide (i.e. $P = Q$) and $\nu_0 = \delta_0$.

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